

Solid Geometry Facts – Solutions and Discussion

Solution 1. If we knew AB, then the volume would be easy! Unfortunately AB is not given to us. But we can figure out AB by using the Pythagorean theorem twice. Here is how.

First, solve for EG, which is one leg of a right triangle whose hypotenuse AG is 24 inches and whose other leg AE is 16 inches. $34^2 - 16^2 = 1156 - 256 = 900 = 30^2$. So EG = 30 inches.

Next, solve for GH, which is one leg of a right triangle whose hypotenuse EG is 30 inches and whose other leg EH = AD = 18 inches. $30^2 - 18^2 = 900 - 324 = 576 = 24^2$. So GH = AB = 24 inches. So:

$$\text{Volume of prism} = \text{AB} \times \text{AD} \times \text{AE} = 24 \times 18 \times 16 = 6912 \text{ cubic inches.}$$

Solution 2. To get the surface area, we just need to compute the areas of the rectangular sides. Be careful not to forget to add opposite sides

$$24 \times 18 = 432 \text{ square inches}$$

$$18 \times 16 = 288 \text{ square inches}$$

$$24 \times 16 = 384 \text{ square inches}$$

$$\text{Sum of all six sides} = 2 \times 432 + 2 \times 288 + 2 \times 384 = 2 \times 1104 = 2208 \text{ square inches.}$$

Solution 3. The area of ADEH is $16 \times 18 = 288$ square inches, and the height GH = 24 inches. The volume of this pyramid is $(1/3) \times 24 \times 288 = 2304$ cubic inches.

Solution 4. The area of ABCD is $24 \times 18 = 423$ square inches, and the height CG = 16 inches. The volume of this pyramid is $(1/3) \times 16 \times 423 = 2304$ cubic inches again.

Solution 5. The last pyramid has rectangular base ABFE which is $24 \times 16 = 384$ square inches and apex G, with height FG = 18 inches. Its volume is $(1/3) \times 18 \times 384 = 2304$ cubic inches again!

You can divide *any* rectangular prism into three rectangular pyramids of equal volume with the same apex like this. How would you divide a **triangular** prism into three **tetrahedral** pyramids of equal volume? [Euclid used that technique to prove the volume a pyramid](#) is one-third the volume of a prism.

Writing Radicals Correctly

Before solving the next question, we need to have a word about how to use the radical (square-root) symbol in math competitions.

Notice that if you have a number like $\sqrt{12}$ you could just as easily write the number as

$$\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$$

If you don't see why this is true, you can check that $2\sqrt{3}$ is the same as $\sqrt{12}$ by squaring it:

$$(2\sqrt{3}) \times (2\sqrt{3}) = 2 \times 2 \times \sqrt{3} \times \sqrt{3} = 4 \times 3 = 12$$

Since there are many ways to write a number with square-root symbols, math team solutions generally require that you write radicals in a canonical form by following two rules:

1. Square-root signs are not allowed in the denominator of a fraction (they can't be on the bottom).
2. Inside a square-root sign you should only use positive square-free integers.

A *square-free integer* is an integer that cannot be divided by a square number larger than 1. For example, 18 is not square-free, nor is 27, because they are both divisible by 9. So you would write:

$$\sqrt{18} = 3\sqrt{2} \text{ , and you would convert: } \sqrt{27} = 3\sqrt{3}$$

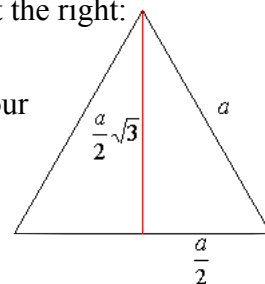
However, 26 is square-free, so you could just leave $\sqrt{26}$ as-is.

On the other hand, if you got the answer $\frac{1}{\sqrt{26}}$, you would have to simplify this to remove the radical from the bottom of the fraction. You can do that by multiplying top and bottom by $\sqrt{26}$ to get:

$$\frac{1}{\sqrt{26}} = \frac{\sqrt{26}}{26}$$

Now, on to problem 6 and beyond which have plenty of radicals that need to be simplified.

Solution 6. The height of an equilateral triangle can be computed as illustrated at the right: just notice that a right triangle is formed where one leg is half the length of the hypotenuse. The other leg will be $\frac{\sqrt{3}}{2}$ times the length of the hypotenuse. In our case, the hypotenuse is a 1 cm edge, so we have:



$$\text{DM is } \frac{\sqrt{3}}{2} \text{ cm (which is about 0.866 cm).}$$

Solution 7. Once we have the height, the area of a face is just $\frac{1}{2} \times b \times h$. The base is 1 cm so area is:

$$\text{Area of a face} = \frac{1}{2} \times b \times h = \frac{1}{2} \times \frac{\sqrt{3}}{2} \times 1 = \frac{\sqrt{3}}{4} \text{ square centimeters (which is about 0.433 cm}^2\text{)}$$

Solution 8. A tetrahedron has four sides, so the total surface area is four times the area of one side, or

$$4 \cdot \frac{\sqrt{3}}{4} = \sqrt{3} \text{ square centimeters.}$$

Solution 9. The triangle CDM has two sides DM and CM each of length $\frac{\sqrt{3}}{2}$ cm, and one longer side CD of length 1 cm. We can calculate the area of CDM by first calculating the height of the isosceles triangle perpendicular to the 1cm base. That height is $\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} - \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

(using the Pythagorean theorem yet again) and so the area of CDM is:

$$\text{Area of CDM} = \frac{1}{2} \times b \times h = \frac{1}{2} \times 1 \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} \text{ square centimeters.}$$

Solution 10. Since we know the area $\frac{\sqrt{2}}{4}$ of CDM = $\frac{1}{2} \times CM \times DO$, and since $CM = \frac{\sqrt{3}}{2}$ centimeters, we have:

$$DO = \frac{\text{Area of CDM}}{\frac{1}{2} \times CM} = \frac{\frac{\sqrt{2}}{4}}{\frac{1}{2} \times \frac{\sqrt{3}}{2}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \text{ centimeters.}$$

DO is not only the height of this triangle; it is also the height of the whole tetrahedron.

Solution 11. The volume of the tetrahedron is:

$$\left(\frac{1}{3}\right) \times h \times A = \left(\frac{1}{3}\right) \times DO \times \text{Area of ABC} = \frac{1}{3} \times \frac{\sqrt{6}}{3} \times \frac{\sqrt{3}}{4} = \frac{\sqrt{2}}{12} \text{ cubic centimeters.}$$

On to cones.

Solution 12. To solve the volume, we need to find r , s , and then h . First, the radius r . If the surface area of the bottom of the cone is 25π square inches, we know that the radius of the circle at the bottom of the cone is 5 inches. Next, the slant height s :

The surface area of the cone is 65π , so

$$65\pi = \pi \times r \times s = \pi \times 5 \times s$$

which means that:

$$s = 65/5 = 13.$$

The surface diagonal s makes a hypotenuse of a right triangle with base $r = 5$. So the height h is:

$$h = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12 \text{ inches.}$$

Now that we have h , we can figure out the volume of the cylinder:

$$\text{Volume} = \left(\frac{1}{3}\right) \times h \times \text{Area} = \left(\frac{1}{3}\right) \times h \times \pi r^2 = \left(\frac{1}{3}\right) \times 12 \times \pi 5^2 = 4 \times 25\pi = 100\pi \text{ cubic inches.}$$

Solution 13. The volume of the cylinder is:

$$\text{Volume} = h \times \text{Area} = h \times \pi r^2 = 1 \times \pi \times 1^2 = \pi \text{ cubic inches.}$$

Its surface area is made of bottom and top circles and the curved part:

$$\text{Curved cylindrical area} = h \times \text{Perimeter} = h \times 2\pi r = 2\pi \text{ square inches.}$$

$$\text{Bottom circle and top circle area} = \pi r^2 = \pi \text{ square inches each.}$$

$$\text{Total surface area} = 2\pi + \pi + \pi = 4\pi \text{ square inches.}$$

Solution 14. The volume of the cone is:

$$\text{Cone volume} = (1/3) h \times \text{Area} = (1/3) h \times \pi r^2 = 1/3 \times \pi \times 1^2 = \pi/3 \text{ cubic inches.}$$

To figure the surface area of the curved part area of the cone, we need to figure the surface diagonal using the Pythagorean theorem. The base (radius) and the height of the triangle are both 1 inch, so the surface diagonal is the hypotenuse that is $\sqrt{1+1} = \sqrt{2}$ inches. So the surface area of the curved part of the cone is:

$$\text{Curved cone area} = \pi \times r \times s = \pi \sqrt{2} \text{ square inches.}$$

$$\text{Bottom circle area} = \pi r^2 = \pi \text{ square inches.}$$

$$\text{Total surface area} = \pi(1+\sqrt{2}) \text{ square inches.}$$

Solution 15. The volume of the shape without the cutout is just the difference of the volumes:

$$\text{Volume} = (\text{Cylinder volume} - \text{Cone volume}) = \pi - \pi/3 = 2\pi/3 \text{ square inches.}$$

The surface area of the shape without the cutout requires careful counting of the parts:

$$\text{Curved cylindrical area} = h \times \text{Perimeter} = h \times 2 \pi r = 2 \pi \text{ square inches.}$$

$$\text{Curved cone area} = \pi \times r \times s = \pi \sqrt{2} \text{ square inches.}$$

$$\text{Bottom circle area} = \pi r^2 = \pi \text{ square inches.}$$

$$\text{Total area} = \pi(3+\sqrt{2}) \text{ square inches.}$$

Solution 16. The outer radius of the ring of cheese is always 1 whole meter, and the area of that outer circle is πr^2 which is just π square meters. The inner radius is x which means the inner circle has area πx^2 . So the area of the ring is the difference between the area of the outer circle and the inner circle, which is just:

$$\text{Area of the ring} = \pi - \pi x^2 = \pi(1 - x^2)$$

On to some polyhedra!

Solution 17. The octahedron can be divided into eight pyramids each with a height of 2 inches and an equilateral triangular base. In the diagram in our solution for question 7 we can see that that the area of an equilateral triangle with side length a is $\frac{1}{2} \times b \times h = a^2 \frac{\sqrt{3}}{4}$. In our case, a is $2\sqrt{6}$, so:

$$\text{Area of one side of the octahedron} = (2\sqrt{6})^2 \times \frac{\sqrt{3}}{4} = 24 \times \frac{\sqrt{3}}{4} = 6\sqrt{3} \text{ square inches.}$$

Since the octahedron has eight congruent faces, we have

$$\text{Surface area} = 8 \times 6\sqrt{3} = 48\sqrt{3} \text{ square inches, about 83.138 square inches.}$$

Each of the eight faces is the base of a pyramid with an apex at the center of an octahedron 2 inches away. The volume of a single such pyramid is:

$$(1/3) \times h \times \text{Area} = \frac{1}{3} \times 2 \times 6\sqrt{3} = 4\sqrt{3} \text{ cubic inches, so:}$$

$$\text{Volume of the octahedron} = 8 \times 4\sqrt{3} = 32\sqrt{3} \text{ cubic inches, about 55.426 cubic inches.}$$

Solution 18. For the dodecahedron we are given the surface area and need to figure the volume.

We can divide the shape into twelve pyramids, each with a pentagonal base and height of 2 inches.

$$\text{Total volume} = 12 \times (\text{volume of a pyramid}) = 12 \times (1/3) \times h \times (\text{Area of a base}).$$

But we are given h which is the distance to the center is 2 inches. And we are given the the surface area of the dodecahedron, which is $12 \times (\text{Area of a base}) \approx$ about 66.6 square inches, as well. So:

$$\begin{aligned} \text{Total volume} &= 12 \times (\text{Area of a base}) \times (1/3) \times h \\ &= 120\sqrt{130-8\sqrt{5}} \times \frac{1}{3} \times 2 = 80\sqrt{130-8\sqrt{5}} \approx 66.6 \times (1/3) \times 2 \approx 44.4 \text{ cubic inches.} \end{aligned}$$

There is a general pattern here:

In general if every face of a 3 dimensional figure is the same distance r to a center point, then we can write the volume as:

$$\text{Total volume} = (1/3) \times r \times (\text{total surface area})$$

Solution 19. This dodecahedron has **12 faces, 20 vertices, and 30 edges**. If we drew a shape connecting all the centers of the faces to the centers of adjacent faces, we would end up with a **regular icosahedron with 20 faces, 12 vertices, and 30 edges**, as pictured in Problem 20.

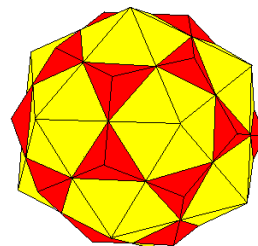
You can check these numbers by noticing that each face has 5 edges, but each edge is shared by two faces. So the number of edges is $12 \times 5 / 2 = 60 / 2 = 30$ edges. Similarly each face has 5 corners, but each corner is shared by 3 faces. So the number of vertices is $12 \times 5 / 3 = 60 / 3 = 20$ vertices.

Solution 20. The reasoning is similar to problem 18.

$$\text{Total volume} = (1/3) \times r \times (\text{total surface area}) = (1/3) \times 60.65 \times (1/3) \times 2 \approx 40.43 \text{ cubic inches.}$$

As you can see, as we make the shape more ball-like, the surface area and volume get smaller.

Solution 21. To get the edges of an icosahedron to cross the edges of a dodecahedron at 90 degrees, the vertices of each shape must be centered over the faces of the other, as pictured at right. The resulting superimposed shape is not convex, but can be wrapped inside a convex **rhombic tricontahedron** with 30 rhombus faces (each centered at a 90-degree edge crossing), 32 vertices (the union of those from the original icosahedron and dodecahedron), and 60 edges.



Solution 22. If we have a polyhedron of volume $160(\sqrt{5}-2)$ cubic inches and inscribed radius 2 inches, then we can use the formula in reverse to figure the surface area:

$$160(\sqrt{5}-2) = \text{Total volume} = (1/3) \times r \times (\text{total surface area}) = (2/3) \times (\text{total surface area})$$

$$\text{Surface area} = \text{Volume} / (2/3) = 160(\sqrt{5}-2) \times (3/2) = 240(\sqrt{5}-2) \approx 56.66 \text{ square inches.}$$

Solution 23. In the diagram, the lines x and y form the legs of a right triangle with hypotenuse 1, since the hypotenuse connects the center of the sphere to the edge. So we can solve for y using Pythagoras:

$$y = \sqrt{1^2 - x^2} = \sqrt{1 - x^2}$$

The area of a circle of radius y is πy^2 which is just:

$$\text{Area of a circle sliced distance } x \text{ from the center} = \pi y^2 = \pi (\sqrt{1 - x^2})^2 = \pi (1 - x^2)$$

Solution 24. All the cheese slices at every distance from the bottom have exactly the same area. You can't get any more cheese with one shape or the other, no matter where you slice it. So:

The two volumes must have the same amount of cheese!

That must mean that the volume of a hemisphere is the same as subtracting the volume of a cone from the volume of a cylinder that has the same radius and height as the hemisphere that we calculated in Problem 15. Of course, that is true not just for radius $r =$ one meter, but any radius:

$$\text{Volume of a hemisphere} = \pi r^3 - (1/3)\pi r^3 = \frac{2}{3}\pi r^3$$

$$\text{Volume of the whole sphere} = \frac{4}{3}\pi r^3$$

After you have solved problem 24, you know why the formula of the volume of the sphere is $\frac{4}{3}\pi r^3$

Solution 25. The volume of a sphere of radius r and surface area A will be the same as the volume of a bunch of pyramids a total surface area A and height r :

$$\text{Volume of pyramids with base } A \text{ and height } r = \frac{1}{3} \times r \times A = \text{Volume of sphere} = \frac{4}{3}\pi r^3$$

But then we can divide both sides by $\frac{r}{3}$ to solve for A :

$$\text{Surface area of the sphere} = \frac{V}{r/3} = \frac{3V}{r} = \frac{3 \times \frac{4}{3}\pi r^3}{r} = 4\pi r^2$$

Now you know why the formula for the surface area of a sphere is $4\pi r^2$.

Solution 26. The curved part of the cylinder in question has surface area $= \pi \times 20 \times 20 = 400\pi$ square inches (be careful not to mix up radius and diameter: the radius of the cylinder is 10 inches). A sphere of radius **10 inches** has the same surface area: $4\pi r^2 = 400\pi$ square inches. A sphere of radius r fits exactly inside the cylinder of radius r and height $2r$ with the same curved surface area as the sphere.

Solution 27. Volume:

$$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi (2)^3 = \frac{32}{3}\pi \text{ cubic inches, about } 33.51 \text{ cubic inches.}$$

Surface area:

$$4\pi r^2 = 4\pi (2)^2 = 16\pi \text{ square inches, about } 50.27 \text{ square inches.}$$

The polyhedra in 17, 18, 20, and 22 get progressively closer to this sphere in surface area and volume.