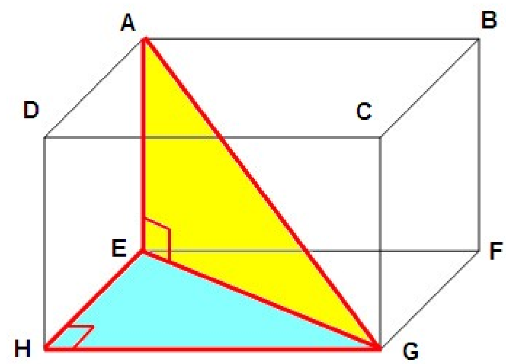
Lincoln Math Team Solid Geometry Facts and 3^3 Problems

PRISM FACT: The volume of a prism = (height of the prism) × (Area of the base)

A rectangular prism is pictured at right.



AD is 18 inches
AE is 16 inches
AG is 34 inches

Problem 1. What is the volume of the prism?

Problem 2. What is the surface area of the prism?

PYRAMID FACT: The volume of a pyramid = <u>1/3</u> × (height of the pyramid) × (Area of the base)

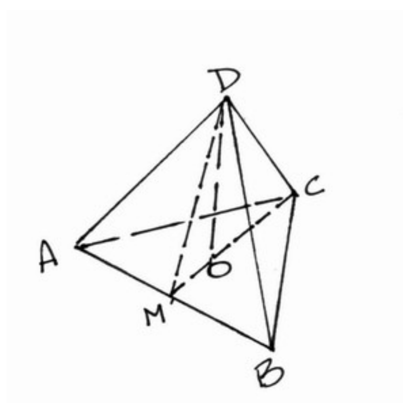
Problem 3. Within the prism above, draw the sideways oblique pyramid with rectangular base ADEH and apex G, so that its height is GH. What is the volume of that pyramid?

Problem 4. Now draw the pyramid with rectangular base ABCD and apex G. What is its volume?

Problem 5. Find a third pyramid that covers all the points in the prism not covered by the pyramids in problems 3 and 4. What is the base of that pyramid and what is the apex? What is its volume?

The base of a pyramid can be any shape and the volume will still be **1/3 × B × h.**

A regular tetrahedron is pictured at right, with AB, BC, AC, CD all equal 1 cm. M is the midpoint of AB, and DOC forms a right angle.



Problem 6. Notice that DM is the height of an equilateral triangle of length 1 cm. What is the length of DM? Please simplify any radicals.

Problem 7. What is the area of the bottom face ABC?

Problem 8. What is the surface area of the tetrahedron?

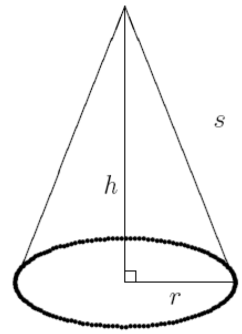
Notice that CDM is an isosceles triangle with DM equal to CM, with base CD and height from M to the midpoint of CD.

Problem 9. What is the area of the triangle CDM?

Problem 10. What is the length of the line DO?

Problem 11. What is the volume of the tetrahedron?

CONES AND CYLINDERS: Just prisms and pyramids with circular bases.



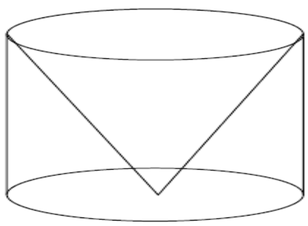
Cylinder: volume = $\boldsymbol{\pi r^2 \times h}$ and surface area of curved part = 2 $\boldsymbol{\pi r \times h}$

Cone: volume = $\boldsymbol{1/3 \times \pi r^2 \times h}$ and surface area of curved part = $\boldsymbol{\pi r \times s}$
(where s is the *slant height* along the slope of the cone, pictured at right)

Problem 12. Suppose the surface area of the curved part of a cone is 65 π square inches and the surface area of the bottom of the cone is 25 π square inches. What is the height of the cone? What is the volume of the cone?

A giant block of cheese with one hole

Pictured below is a huge cylindrical block of cheese of height 1 meter with a circular base of radius 1 meter also. A single cone of height 1 and the same circular base has been removed from it.



Problem 13. What are the the volume and surface area of the whole cylinder including the top and bottom?

Problem 14. What are the volume and surface area of the cut out cone including the top?

Problem 15. What are the the volume and surface area of the shape that remains after after the cone is removed from the cylinder including the bottom circle?

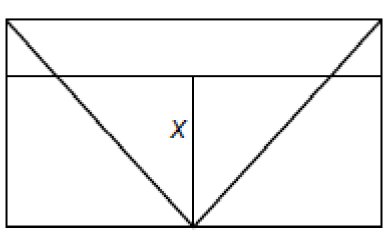
Write down the volume of this strange cut-out block of cheese. We will come back to it in Problem 24.

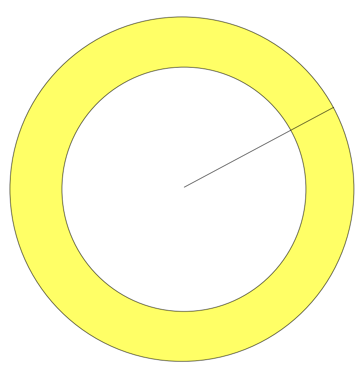
CROSS-SECTIONAL AREA

If you slice through a 3d object, the slice will make a 2d shape with an area.

Let us make a horizontal slice through the cheese block and calculate the area of the cross-section. Any cross-section through this block of cheese will be a ring.

Problem 16. If you measured x meters up from the bottom of the block of cheese and sliced the block of cheese at that point, what would be the area of the flat ring of cheese at the level of the slice. The answer should be in square meters in terms of x.

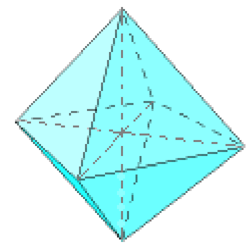




POLYHEDRA: Figure the volume by dividing into pyramids

The volume of a polyhedron that is not a pyramid can be computed by dividing it into pyramids and then adding up volumes.

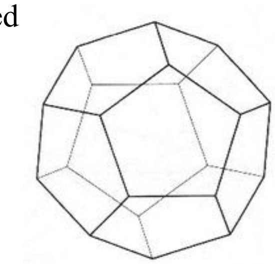
Pictured at right is a regular octahedron whose edges are all equilateral triangles with edge length $2\sqrt{6}$ inches. Each triangular face has area $6\sqrt{3}$ square inches (about 10.39 square inches). The octahedron can be subdivided into eight congruent triangular pyramids that meet with all their apexes at the center, as shown by the dotted lines.



The distance from the center of one of the triangular faces to the center of the whole octahedron is 2 inches. (Another way to say this is that a sphere of radius 2 inches can be inscribed in this octahedron.)

Problem 17. What is the surface area and volume of the whole octahedron?

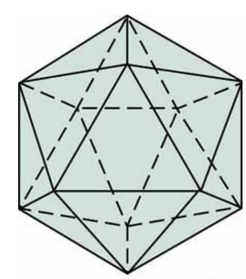
Now suppose you have a regular dodecahedron that fits tightly around an inscribed sphere of radius 2 inches. (A regular dodecahedron is a 12-sided figure with regular pentagon faces, as pictured at right.) It will have a total surface area of $120\sqrt{130-8\sqrt{5}}$ ≈ 66.6 square inches.



Problem 18. What is the approximate volume of this dodecahedron in cubic inches, rounded to the nearest tenth of a cubic inch?

Problem 19. How many edges and vertices (corners) are in this dodecahedron? What shape would you get if you drew an edge between the center of every face of the dodecahedron to the center of the five adjacent faces? How many faces, edges, and vertices would that polyhedron have?

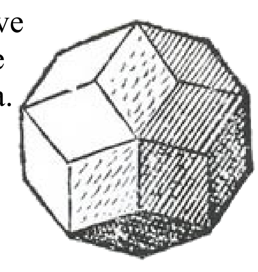
The 20-sided figure pictured at right is called a regular icosahedron. Suppose you have one that fits tightly around an inscribed sphere again of radius 2 inches. It will have a surface area of $840\sqrt{3}-360\sqrt{15}$ ≈ 60.65 square inches.



Problem 20. What is the volume of this icosahedron in cubic inches, to the nearest tenth of a cubic inch?

Problem 21. Find a way to superimpose a regular dodecahedron with a regular icosahedron in space so that each edge of the icosahedron meets a single edge of the dodecahedron at a 90 degree angle. What is the smallest convex polyhedron that contains both superimposed shapes? How many faces, edges, and vertices does that polyhedron have?

The regular tetrahedron, octahedron, dodecahedron, icosahedron and cube are the five **Platonic solids**, named after the ancient Greek philosopher who revered them. More recent mathematicians have discovered a number of other very symmetric polyhedra. The 30-faced **rhombic triacontahedron** at right was discovered by Kepler in 1611.

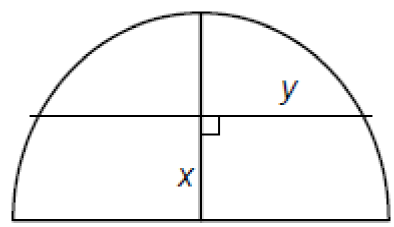


Problem 22. A rhombic triacontahedron that has an inscribed sphere of of 2 inches has **volume** $160(\sqrt{5}-2)$ ≈ 37.77 cubic inches. What is its **surface area**?

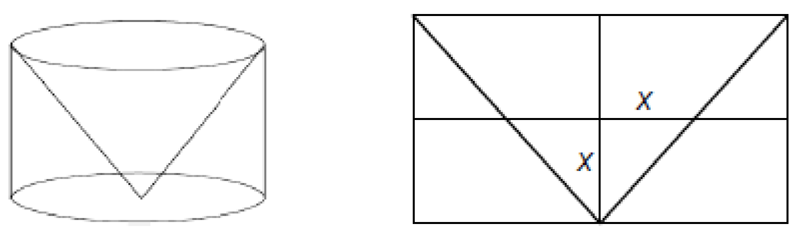
SPHERES: There are two formulas to memorize and understand

Suppose you have a hemisphere of radius one meter made out of cheese. We will take a look at some slices to try to decide how the amount of cheese compares to the shape in problem 16.

Problem 23. If you measure a distance x from the center of the sphere and then slice the cheese hemisphere at that distance from the center, the cross section will be a circle. What is the radius y of that circular slice at height x, in terms of x and r? What is the area of the cross-sectional circle?



Remember from problem 16 the details of the other shape with cheese slices. There the picture looked like this:



Problem 24. How do you think the **volume** of the solids from problems 16 and 23 compare given that the radius and height is one meter in both cases? Which solid has more cheese, the one-meter hemisphere, or the cylinder of radius and height one meter with the cone of radius and height one meter removed from it? Using your answer to problem 15, can you infer the volume of the hemisphere?

There are two formulas for spheres that need to be memorized:

VOLUME OF A SPHERE of radius r = $(4/3)\pi r^3$

Problem 25. If the volume of a sphere is V and the radius to the center is r, then write a formula for the surface area of the sphere in terms of V and r. Think about your answer to problem 22.

SURFACE AREA OF A SPHERE of radius r = $4\pi r^2$

Problem 26. What is the radius of a sphere that has exactly the same surface area the curved part of a cylinder of height 20 inches and diameter 20 inches? Would that sphere fit inside the cylinder?

Problem 27. What is the exact volume and surface area of a sphere of radius 2 inches? How does this compare to your answers to problems 17, 18, 20, and 22?